Optimal monetary policy under heterogeneous beliefs of the central bank and the public

Alex Ilek and Guy Segal

Abstract

If the central bank and the public have different beliefs concerning the structure of the economy, we argue that the central bank should form its policy in a two-stage process: (1) it should model the expectations formation of the public; (2) it should combine the public's model along with its own structural model and only then derive the optimal policy rule. Using Monte-Carlo simulations, we show that this approach improves welfare significantly compared with a case in which the central bank reacts to the public's expectations but ignores the expectations formation mechanism.

Keywords: heterogeneous beliefs; optimal monetary policy; expectations formation; learning; welfare

JEL codes: E52, E58, E61.

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1 Introduction

The mainstream literature of optimal monetary policy (hereinafter, OMP) assumes rational expectations (hereinafter, RE). According to the assumption of RE, both the central bank (hereinafter, CB) and the public possess full information concerning the same model of the economy, and therefore their expectations are consistent with each other (e.g., Clarida, et al., 1999, Woodford, 2003a, and Walsh, 2010).

However, in recent years a growing body of literature has focused on OMP when the public have different perceptions from the CB concerning the economy.

In this setting, we argue that to make monetary policy efficient, the CB should learn the public's expectations formation mechanism, utilize it along with its own model, and only then derive the optimal policy rule. This approach requires the modeling of the expectations formation mechanism of the public, but the implementation itself is a relatively simple task. We find that applying this approach results in a considerable welfare gain, as the CB exploits the contemporaneous and dynamic relationships between its own model and the public’s model, irrespective of whether the public's model is correct or not.2

Berardi and Duffy (2007) and Cone (2005) assumed that agents misspecify the model of the CB. They examined the value of the transparency of the CB: In the transparent regime, the CB announces its model and objectives and thus enables the public to adopt the CB’s model; under a nontransparent regime the public keep underspecifying the CB's model. Berardi and Duffy (2007) concluded that in the discretion case the advantage of transparency is not unequivocal and depends on policy target values of inflation and output. Cone (2005) found that transparency of

2 The CB can also exploit the dynamics of the parameters via the learning algorithm of the public. Vestin, et al. (2006a, 2006b), and Molnar and Santoro (2010) examined this channel of monetary policy in a very parsimonious framework and the main challenge is to apply it to richer models.
the CB with respect to its model is desirable only if inflation expectations of the public are far away from RE. According to Cone (2005), an announcement of the CB's model would shift the perceptions of the public toward the perceptions of the CB and would stabilize the economy and increase the welfare gain.

In contrast to Berardi and Duffy (2007) and Cone (2005), we assume that the discrepancy in beliefs between the CB and the public cannot be eliminated, even by a fully transparent and communicative CB. A prolonged or even permanent discrepancy between the CB and the public with respect to the economy can occur: (1) if the CB's short to medium horizon forecasts suffer from limited credibility; (2) if there is a practical inability of the public to adopt the model and the perceptions of the CB.

A necessary condition, but not a sufficient one, for shifting the public’s perceptions to be in line with the CB's is that the CB is transparent with respect to its models and its forecasts. The sufficient condition requires full credibility of the CB: the public should believe that the models and projections of the CB are better than its own. However, in practice, there is no convincing evidence for such behavior. Although the CB announces its projections of the economy and posts its models to the public on a regular basis, capital market participants, professional forecasters and households seem to perceive the economy differently than the CB by reporting different projections. One possible explanation is that agents may think that although the CB may have a rich and complicated theoretical model, it does not necessarily provide better forecasts than their own forecasts. Thus, there is a subjective probability of the public that its own model is better. Agents may also doubt the CB’s projections if they think that even if the CB has an incorrect model of the economy, it has an incentive to convince the public to adopt its own model.
and projections of the economy (principle-agent problem). A potential reluctance of
the public to adopt the model of the CB (and its forecasts), which has nothing to do
with credibility, may stem from the fact that the CB's models are generally
complicated and hardly operational. Another technical difficulty arises from the fact
that the projections of the CB are published on a low frequency basis and hence are
not useful for high frequency usage.

A survey of professional forecasters in Israel supports these two arguments, and
thus supports the assumption of discrepancy in perceptions of the economy between
the CB and the public (Appendix D).

Adam and Woodford (2012) considered the case of different models of the CB
and the public. They proposed an approach for designing OMP where agents’ beliefs
are close to satisfying RE but the exact formation of their beliefs is not known to the
CB. They showed that in this setting, the long-term inflation rate underreacts to the
cost push shock. ³ Their overall conclusion, however, is that the policy prescriptions
arising under the assumption of RE of the public are robust in an environment of near-
ratational expectations. In contrast to the aforementioned approach, we do not rely on
the restrictive assumption that expectations of the public are in the neighborhood of
RE—rather, they could take any form.

Evans and Honkapohja (2003) proposed an expectations-based policy rule, which
is derived under the assumption that agents have RE, and it is expressed in terms of
the public’s expectations. Therefore, the Evans and Honkapohja rule could be used for
any type of expectations. Berardi (2009) exploited such a rule (under a timeless
perspective regime) in a case where a subgroup of agents has a correct model,
whereas the other subgroup has an underspecified model. Berardi concluded that the

³The authors applied a commitment regime of the CB. Under the robust policy rule, the initial response
of inflation to cost-push shock is smaller than implied under RE, but in the subsequent periods the
dynamics are the same.
CB should consider only the correct expectations and disregard the misspecified ones. Berardi's conclusion stems from an assumption that the misspecified expectations are nested in the correct expectations; therefore, when the CB responds to the correct expectations it actually responds to the misspecified expectations as well. Thus, Berardi's conclusion may not hold when a subgroup of the public have expectations which are not nested in the model-consistent expectations.

Gasteiger (2014) tested an Evans and Honkapoja type rule where a share of the public has RE and the rest have naïve, adaptive or extrapolative expectations. He showed that this rule is robust to heterogeneity of the public's expectations with respect to determinacy. However, the question of which policy rule better reduces the fluctuation in the economy until convergence to the RE equilibrium, which is at the heart of our paper, was beyond the scope of Gasteiger (2014).

Orphanides and Williams (2008) showed that when the public forms its expectations using a VAR model estimated by a constant gain learning algorithm, the CB policy that is derived under an erroneous RE assumption "can perform poorly", with respect to welfare, relative to the case where agents have RE. To reduce the welfare loss, Orphanides and Williams (2008) proposed a policy rule which has the same specification as the rule derived under RE. The only difference is that the parameters are chosen to minimize the loss function subject to the public's expectations derived from the VAR model. Thus, the approach proposed by Orphanides and Williams (2008) is different than our approach in one fundamental way: in their approach, the policy rule is derived under the RE assumption and only at the second stage are adjustments made to accommodate it to the specific expectations formation of the public. In our approach, the derivation of the policy rule itself—both
its specification and parameterization—takes into account, already at the first stage, the discrepancy in beliefs between the CB and the public.

The rest of the paper is organized as follows: Section 2 presents the canonical New Keynesian model of the central bank. Section 3 presents the model of the public. Section 4 proposes treatment of the expectations formation of the public. Section 5 presents the model calibration. Section 6 explains the various learning frameworks of the public. Section 7 presents welfare analysis and Section 8 concludes.

2 The Model of the Central Bank

The CB uses the canonical New Keynesian model of Clarida, et al. (1999). The model consists of two behavioral equations:

2.1 The Phillips curve (inflation equation)

\[ \pi_t = \beta \pi_{t-1} + k x_t + \varepsilon_t^\pi, \]  \hspace{1cm} (1)

where \( \pi_t \) is the inflation rate, and \( x_t \) is the output gap which is defined as the gap between actual output, \( y_t \), and potential output. For simplicity, we assume that the potential output is constant at the level of the output in the steady state, \( y \). \( \varepsilon_t^\pi \) is a white-noise shock to inflation. \( E_t \pi_{t+1} \) denotes inflation expectations of the public. All variables are expressed as deviations from the steady state values.

2.2 The demand equation (IS equation)

\[ x_t = E_t x_{t+1} - \alpha (i_t - E_t \pi_{t+1}) + \varepsilon_t^x, \]  \hspace{1cm} (2)

where \( i_t \) is the interest rate of the CB and \( E_t x_{t+1} \) denotes the public's expectations for the output gap in the next period. \( \varepsilon_t^x \) is a white-noise demand shock.
In a matrix form, equations (1) and (2) are given by:

\[ z_t = F_1 z_{t-1} + F_2 i_t + F_3 \varepsilon_t, \]  

(3)

where \( z_t = [\pi_t, x_t]' \), \( \varepsilon_t = [\varepsilon_t^\pi, \varepsilon_t^x]' \), \( F_1 = \begin{bmatrix} \alpha k + \beta & k \\ \alpha & 1 \end{bmatrix} \), \( F_2 = \begin{bmatrix} \kappa \alpha \\ -\alpha \end{bmatrix} \), \( F_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \).

Equation (3) is the state representation of the economy expressed in terms of the public’s expectations (which are not necessarily rational), the policy rule and the shocks. To close the model, the monetary policy should be added.

2.3 Monetary Policy

We assume that the CB acts in an optimal discretionary manner and minimizes the standard quadratic loss function:

\[ E_t L_t = E_t \sum_{j=0}^{\infty} \beta^j [\pi_{t+j}^2 + \lambda x_{t+j}^2]. \]  

(4)

\( \lambda \) is the relative weight of the output gap with respect to inflation in the loss function and \( \beta \) is the discount factor. We assume that the CB holds the correct structural model of the economy.

3 The Model of the Public

For simplicity, we assume that the public's expectations are formed by a VAR model which the CB observes. Furthermore, we assume that the VAR model consists of three variables (with one lag): inflation, interest rate and an indicator of real activity. Specifically, following Branch and Evans (2006), Loungani (2001), Oller and Barot (2000) we assume that the public uses the growth rate of output as the indicator of real activity. This assumption seems highly plausible: most, if not all professional forecasters and surveys worldwide report their projections for the output growth rate
rather than for the output gap, and it is most unlikely that these projections are consistent with the output gaps derived from the CB's model. The same picture is depicted by a survey we conducted among Israeli professional forecasters (Appendix D). The VAR model of the public is given by:

$$\begin{bmatrix} \tilde{z}_t \\ i_t \end{bmatrix} = A \begin{bmatrix} \tilde{z}_{t-1} \\ i_{t-1} \end{bmatrix} + u_t,$$

(5)

where $\tilde{z}_t = [\pi_t, \Delta y_t]'$ is a vector consisting of the inflation rate and the growth rate of the output, respectively. $u_t$ is a $3 \times 1$ matrix of the residuals.

Next, we will concentrate on the submatrix $\tilde{z}_t$ and rewrite it as:

$$\begin{bmatrix} \pi_t \\ \Delta y_t \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{bmatrix} \begin{bmatrix} \pi_{t-1} \\ \Delta y_{t-1} \\ i_{t-1} \end{bmatrix} + \tilde{u}_t.$$

(6)

We will express the VAR model in terms of the CB’s structural model. Specifically, the growth rate of the output should be represented in terms of the output gap.

Assuming that the potential output is constant (for simplicity):

$$\Delta y_t = y_t - y_{t-1} = y_t - y - (y_{t-1} - y) \equiv \Delta y_t,$$

(7)

the VAR model in terms of the output gap is given by:

$$\begin{align*}
\pi_t &= A_1 \pi_{t-1} + A_2 x_{t-1} - A_3 x_{t-2} + A_3 i_{t-1}, \\
x_t &= B_1 \pi_{t-1} + (1 + B_2) x_{t-1} - B_3 x_{t-2} + B_3 i_{t-1}.
\end{align*}$$

(8)

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4 A complete analysis of the VAR model of the public expressed in terms of the output gap can be provided upon request.
5 We ignore the last equation for the interest rate because in this framework it does not play any role in the subsequent analysis.
6 We omit residuals since they do not play any role in the subsequent analysis.
Iterating the system of equations (8) one period ahead and taking expectations, we get:

\[ E_t z_{t+1} = S_1 z_t + S_2 x_{t-1} + S_3 i_t, \]  

where \( S_1 = \begin{bmatrix} A_1 & A_2 \\ B_1 & 1 + B_2 \end{bmatrix}, \ S_2 = \begin{bmatrix} -A_2 \\ -B_2 \end{bmatrix}, \ S_3 = \begin{bmatrix} A_3 \\ B_3 \end{bmatrix}. \)

Equation (9) expresses the public's model in terms of the structural model of the CB.

4 Monetary policy under different treatments of the public's expectations

The starting point for our analysis is that there is a difference between the CB's model and the public’s model. We consider two main cases: Case A considers the standard approach in the literature where the CB uses the expectations-based policy rule \textit{a la} Evans and Honkapohja (2006). In that case, monetary policy uses the optimal discretionary policy rule and responds to the public's expectations without modeling their formation; case B considers the proposed approach where the CB derives its policy rule after exploiting the expectations formation mechanism of the public and combining it with its own model. We also consider two alternatives regarding the parameters in the VAR model of the public: updating them each period (learning) or keeping them fixed (not learning).

4.1 Case A - The CB responds to the public's expectations without modeling them

We follow Evans and Honkapohja (2006) and represent the optimal discretionary interest rate rule in terms of the public's expected inflation and expected output gap.
Using (1), (2) and the optimal discretionary first-order condition \( k\pi_t + \lambda x_t = 0 \) derived by Clarida, et al. (1999), the policy rule can be described as:

\[
i_t = (1 + \frac{\beta k}{\alpha(k^2 + \lambda)})E_t\pi_{t+1}^e + \frac{1}{\alpha} E_t x_{t+1}^e + \frac{k}{\alpha(k^2 + \lambda)} \varepsilon_t^\pi + \frac{1}{\alpha} \varepsilon_t^x. \tag{10}
\]

We denote this expectations-based policy rule as EHR (Evans-Honkapohja rule). Although the policy rule (10) is derived under RE assumption, it could be applied for any type of expectations, and Evans and Honkapohja expressed it in terms of the public's expectations. The EHR has been advocated in the literature as a robust rule to any type of public expectations (Evans and Honkapohja (2006a, 2006b), Berardi and Duffy (2007), Berardi (2010) and Gasteiger (2014)). Hence, finding a better rule is a great challenge.

If the CB completely ignores the discrepancy in beliefs between it and the public and acts according to its model-consistent expectations optimal response, then the optimal discretionary policy rule is given by (in the case of no serial correlation in the shocks):

\[
i_t = \frac{k}{\alpha(k^2 + \lambda)} \varepsilon_t^\pi + \frac{1}{\alpha} \varepsilon_t^x \tag{11}
\]


Evans and Honkapohja (2006) showed that if the CB follows this rule when the public’s expectations are boundedly rational\(^7\), then this rule leads to divergence of the economy and to a large welfare loss. Thus, ignoring the public’s expectations in the policy rule is extremely costly, and therefore we do not consider it here.

\(^7\) Bounded rational agents know the correct specification of the model but not its parameters and they learn them over time.
To derive the solution of the economy under the EHR we first plug (10) into (3).

This gives us the following:

\[
\pi_t = (\beta - \kappa \alpha \mu) E_t \pi_{t+1} + \frac{\lambda}{k^2 + \lambda} \pi_{t+1},
\]

\[
x_t = -\alpha \mu E_t \pi_{t+1} - \frac{k}{k^2 + \lambda} \pi_{t+1}.
\]

Note that at this stage we haven’t yet specified the expectations formation of the public, therefore (12) holds for any type of expectations. To derive the solution of the model in terms of lagged variables and current shocks we need to solve for \( E_t \pi_{t+1} \) in (12). Taking the VAR model of the public one period ahead and applying conditional expectations for \( t + 1 \), together with (12) yields (Appendix B):

\[
z_t = \Phi_1 x_{t-1} + \Phi_2 \pi_{t+1},
\]

where the parameters \( \Phi_1, \Phi_2 \) are given in Appendix B.

The parameters \( \Phi_1, \Phi_2 \) are time variant as long as the parameters in the VAR model of the public are time variant, that is, when the public is learning. Equation (13) indicates that both the inflation and the output gap are affected not only by the current shocks but also by the lagged output gap. While a dependence on the historical output gap also exists under the canonical commitment regime, the source of this dependence here is completely different and stems from the specification of the VAR model of the public.
4.2 Case B - Optimal monetary policy when the Central Bank considers the public's expectations formation

Here, the CB learns the model perceived by the public and exploits this information in deriving the optimal rule. Inserting the public’s expectations in (9) into (3) yields:

\[ z_i = F^*_1 S_1 z_i + F^*_1 S_2 x_{-1} + (F^*_1 S_3 + F^*_2) i_i + F^*_3 e_i. \]  

Equation (14) points to the existence of a simultaneity problem: The realization of the inflation rate, the output gap and the interest rate depends on the public's expectations, which in turn, depend on the realization of the economy at the current period. The solution of the system will determine the fixed point of this simultaneity problem. Moving \( F^*_1 S_1 z_i \) to the left hand side of (14), the state space representation of the economy implied by the public's expectations is given by:

\[ z_i = \Lambda_1 x_{-1} + \Lambda_2 i_i + \Lambda_3 e_i, \]  

where \( \Lambda_1 = (I - F^*_1 S_1)^{-1} F^*_1 S_2 \), \( \Lambda_2 = (I - F^*_1 S_1)^{-1} (F^*_1 S_3 + F^*_2) \), \( \Lambda_3 = (I - F^*_1 S_1)^{-1} F^*_3 \).

Now the CB has to derive the optimal policy rule which minimizes the expected loss function (4) subject to the constraint in (15). Due to the existence of the lagged output gap in (15), the analytical solution for finding the optimal policy rule is very cumbersome. This is because there is an endogeneity problem: expected losses depend on the reduced form solution of the economy, which is unknown unless the policy rule is determined. At the same time, the determination of the policy rule depends on the expected losses, which depend on the reduced form solution of the economy.
Dennis (2007) proposed a numerical method to resolve this kind of problem. Following Dennis from (15) and based on the minimum state variable (hereinafter MSV) solution, we observe that the specification of the optimal policy rule is based on the state variables, such that the interest rate will react to the lagged output gap and the current shocks. For simplicity, from now on we will denote this optimal policy rule as POR (Proposed Optimal Rule). The POR is given by:

\[ i_t = H_1 x_{t-1} + H_2 \epsilon_t, \]  

(16)

where \( H_1 \) and \( H_2 \) are obtained numerically using Dennis’s (2007) method.8 Plugging the POR in (16) into (15) we get the actual law of motion (hereinafter, ALM) of the economy implied by the VAR model of the public and the POR:

\[ z_t = \Psi_1 x_{t-1} + \Psi_2 \epsilon_t, \]  

(17)

where

\[ \Psi_1 = \Lambda_1 + \Lambda_2 H_1, \quad \Psi_2 = \Lambda_3 + \Lambda_2 H_2, \quad \Lambda_1 = (I - F_1 S_1)^{-1} F_1 S_2, \quad \Lambda_2 = (I - F_1 S_1)^{-1} (F_1 S_3 + F_2), \quad \Lambda_3 = (I - F_1 S_1)^{-1} F_3 \]

and \( H_1, H_2 \) are obtained from the numerical simulations.

Equation (17) reveals that the specification of the solution under the EHR is equivalent to that under the POR (equation 13): both inflation and the output gap react to the lagged output gap and to current shocks. The difference is reflected in the elasticities—while the solution in (17) minimizes the welfare loss of the economy, the solution in (13) does not (Section 7).

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8 The Dynare and Matlab codes for derivation of the POR are available upon request.
5 Calibration

The structural parameters of the CB's model (including the standard deviations of the shocks) are calibrated following Woodford (2003b) (Table 1).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda )</td>
<td>0.05 (^9)</td>
<td>Weight of the output gap in the objective function</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.99</td>
<td>Discount factor</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>0.0238</td>
<td>New Keynesian Phillips Curve slope</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>1/0.1571</td>
<td>Intertemporal elasticity of substitution in consumption (inverse)</td>
</tr>
<tr>
<td>( \sigma^\delta )</td>
<td>0.005</td>
<td>Standard deviation of the cost-push shock</td>
</tr>
<tr>
<td>( \sigma^\xi )</td>
<td>0.0071</td>
<td>Standard deviation of the demand shock</td>
</tr>
</tbody>
</table>

We follow McCallum and Nelson (2004) to calibrate the standard deviation of the cost-push shock. We calibrate the variance of the demand shock to be twice the variance of the cost-push shock.\(^{10}\) The constant gain learning parameter (for both inflation and output growth equations) is set to 0.02, similar to Orphanides and Williams (2008), Branch and Evans (2006) and Molnar and Santoro (2014). The calibration is fitted to quarterly data, so we treat "periods" in the Monte-Carlo simulations below as quarters.

6 Learning frameworks of the public

Section 3 showed that the VAR model of the public (henceforth PLM, Perceived Law of Motion) is given by:

\[
\tilde{z}_t = AW_{t-1} + u_t, \tag{18}
\]

where \( \tilde{z}_t = [\pi_t, \Delta y_t]' \), \( W_{t-1} = [z_{t-1} \ i_{t-1}]' \), and \( A \) is the matrix of the parameters.

\(^9\) Woodford (2003b) calibrated \( \lambda = 0.048 \).

\(^{10}\) McCallum and Nelson (2004) calibrated the standard deviation of the demand shock to be about 8 times higher than the standard deviation of the cost-push shock. This calibration strengthens our results (Section 7).
6.1 Prior formation
For tractability, we assume that there is a pre-sample period, before agents learn, where data is generated by the REE. In the first period of learning, agents set their priors as RE with some error. Technically, to create such an error in the public's priors we assume that agents exploit a relatively small sample size of $T=50$ periods. This creates a small sample bias in the VAR estimated parameters relative to the parameters implied by the REE, leading to some deviation of the public's expectations from RE.

6.2 The learning frameworks
We test three alternatives relating to the expectations formation mechanism of the public: decreasing gain learning, constant gain learning and non-learning. We conduct Monte-Carlo simulations to analyze the consequences of using either POR or EHR by the CB on welfare, as is denoted by the loss function in (4). The welfare depends on the parameters matrix $A$ in (18), which depends on the learning framework of the public.

6.2.1 Decreasing gain learning agents
Agents update the parameters in their VAR model using decreasing gain learning algorithm (Evans and Honkapohja, 2001)\(^1\). This learning process is given by:

$$A_t = A_{t-1} + t^{-1} R_t^{-1} W_{t-1}(z_t - A_{t-1} W_{t-1})$$

$$R_t = R_{t-1} + t^{-1} (W_{t-1} W_{t-1}' - R_{t-1})$$

where $R_t$ is a matrix of second moments and $t^{-1}$ is a decreasing gain over time.

\(^1\) If agents are uncertain about the stochastic structure of the economy they can use the endogenous gain (see Ilek, 2013).
The decreasing gain learning framework reflects the assumption that the structure of the economy is stable. Appendix C shows that when the CB exploits either POR or EHR and agents learn the parameters in the VAR using the decreasing gain learning algorithm, the economy eventually converges to the MSV solution of REE.

6.2.2 Constant gain learning agents

The constant gain learning is a very popular learning tool in the literature since its assumptions are realistic. This learning process is given by:

\[
A_t = A_{t-1} + g R_{t-1} W_{t-1} (z_t - A_{t-1} W_{t-1})
\]

\[
R_t = R_{t-1} + g (W_{t-1} W_{t-1}' - R_{t-1}), \quad \text{where } g \text{ is a constant gain parameter.}
\]

The constant gain learning is more robust than the decreasing gain algorithm for an economy which experiences frequent structural changes. The constant gain framework weights recent data more heavily than past data (Evans and Honkapohja (2001), Orphanides and Williams (2008), Branch and Evans (2006)). Hence, in the constant gain learning, convergence to REE is never obtained (Evans and Honkapohja, 2001). Although we assume that the structure of the economy, as reflected in the New Keynesian model, is stable, agents may perceive it differently and hence choose learning with a constant gain.

6.2.3 Non-Learning agents

We also analyze the case where agents do not update their priors over time, namely:

\[
A_t = A_{prior} \quad \forall t.
\]

Constant parameters over time can reflect not only potential reluctance of agents to learn (and stick to their priors for various reasons), but also a high degree of
frictions in the learning process. These frictions could be justified, for example, by high information costs or long calculation time (Branch 2004), such that the models are re-estimated on a low frequency basis.

Note that when the CB acts either with POR or EHR, given that the parameters in the VAR model of the public are fixed, the parameters $\Psi_1, \Psi_2$ and $\Phi_1, \Phi_2$ in the solution under POR and in the solution under EHR are fixed as well (equations 13 and 17). Moreover, as long as the public believes that the lagged output growth should be included in the VAR model, the ALM of the economy depends on the lagged output gap, namely $\Psi_1 \neq 0$. Hence, as in the constant gain learning, the economy systematically deviates from the RE equilibrium.

To summarize, the economy eventually converges to REE both under the POR and EHR when agents are learning with decreasing gain. Hence, there is no difference between POR and EHR with respect to the welfare of the economy in the long-run. However, in the short to medium run, there could be a noticeable discrepancy in beliefs between the public and the CB, which might cause a significant welfare loss. We test this in the next Section.

7 Welfare analysis

7.1 Monte-Carlo simulations

The welfare analysis is made using Monte-Carlo simulations with 5,000 replications. In each replication there are two steps:

1) In each simulation, agents form their priors of the parameters in the VAR model based on the pre-sample data (Section 6.1).
2) The main sample consists of 60 periods (15 years of quarterly data). In each period new shocks hit the economy and the CB utilizes either the POR or the EHR. At the same time, agents either learn (update the VAR parameters as the new outcome of the economy is realized) or keep forming their expectations using the parameters fixed. This process continues from period T=1 up to T=60.

7.2 Results of Monte-Carlo simulations

Table 2 presents the expected discounted loss per quarter under the POR and the EHR\textsuperscript{12} in the first 60 quarters, among the three tested alternatives with respect to the agents' behavior—decreasing gain learning, constant gain learning and non-learning agents. The last column presents the ratio between the expected loss (per quarter) under the EHR and under the POR. The higher the ratio (relative to 1), the more efficient the POR is relative to the EHR.

The simulation results show that in all three alternatives of expectations formation, the welfare loss obtained by the EHR is noticeably higher than the loss under the POR—the POR is a robust rule which reduces welfare loss. Specifically, under the EHR, the loss is higher by 8-9\% than under the POR when agents are learning and by 29\% when agents are not learning.\textsuperscript{13} The main reason for this robustness is that with the POR, the CB accommodates the potentially harmful effects of the public expectations, both on current and future dynamics of the economy. For example, historically high output growth leads, according to the public's perceptions, to high expected output growth. As these expectations directly affect the economy, this would lead to a higher output growth and higher inflation,

\textsuperscript{12} Expected discounted loss was calculated based on 60 periods, as explained in Section 6.1.

\textsuperscript{13} When the standard deviation of the demand shock is 8 times higher than the cost-push shock, as was calibrated by McCallum and Nelson (2004), the loss ratio is slightly above 1.3 in all three learning alternatives.
which will increase the welfare loss. This knowledge is useful for the CB, because it can adjust the interest rate to offset this negative effect of the public's perceptions on the economy.

Table 2
Comparison of expected loss per period under POR versus EHR

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<thead>
<tr>
<th></th>
<th>Expected loss</th>
<th>Loss Ratio:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1) Under POR</td>
<td>(2) Under EHR</td>
</tr>
<tr>
<td>Decreasing gain learning public</td>
<td>0.189%</td>
<td>0.205%</td>
</tr>
<tr>
<td>Constant gain learning public</td>
<td>0.183%</td>
<td>0.201%</td>
</tr>
<tr>
<td>Non-learning public</td>
<td>0.218%</td>
<td>0.283%</td>
</tr>
</tbody>
</table>

Table 2 shows that the per-period loss in the first 60 periods is similar, whether agents learn with a decreasing gain or a constant gain, but it is noticeably higher when agents do not learn. The main reason for this result is that when agents do not learn but stick to their priors (which could be far away from RE), their perceptions permanently destabilize the economy, as they induce systematic gaps between the CB's model and their own perceptions, despite the efforts of the CB to mitigate this negative effect. Note that when agents learn with decreasing gain learning algorithm, the advantage of the POR relative to EHR diminishes over time, because the economy converges to the REE implying POR and EHR are identical.

8. Conclusions

This paper proposes a novel approach to designing optimal monetary policy when the CB and the public use different models of the economy. The proposed framework requires the CB to act in a two-stage process: first, it should learn the expectations formation of the public. Second, the CB should incorporate the public’s expectations
formation into its own structural model and only then derive the optimal policy rule. It is important to emphasize that the need to learn the public’s model arises irrespective of whether the CB agrees with such a model or not. This is under the plausible assumption that the public's expectations directly affect the evolution of the economy, irrespective of how accurate these expectations are or their source.

To illustrate the benefits of the proposed approach, we use a framework, where the CB uses a simple structural New Keynesian model and fully observes the VAR model of the public. We show that when the CB exploits the expectations formation of the public instead of using the bank's expectations-based rule, the expected loss is noticeably reduced.

We are aware that modeling the expectations formation of the public is always subject to misspecification and errors, which could potentially reduce the advantage of the proposed framework for monetary policy. Nevertheless, we believe that once a reasonable model of the expectations formation of the public is obtained, the CB can achieve its goals more efficiently using the proposed framework.

There are several main challenges for future research: (1) To implement the proposed framework in richer models, and (2) to derive the optimal policy rule which exploits not only the expectations formation of the public with respect to the variables, but also with respect to the parameters which vary over time as a result of the learning process of the public.
Abbreviations

ALM - Actual Law of Motion
CB - Central Bank
HER - Evans-Honkapoja Rule
MSV - Minimum State Variable
OMP - Optimal Monetary Policy
PLM - Perceived Law of Motion
POR - Proposed Optimal Rule
RE - Rational Expectations
REE - Rational Expectations Equilibrium

References


Appendix A
Derivation of the dynamics of the economy in terms of the VAR model of the public

This appendix expresses the ALM of the economy in terms of the PLM of the agents. We start with adding and subtracting \( x_{\tau-2} \) from the ALM in (17):

\[
\begin{bmatrix} \pi_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} \Psi_1 \\ \Psi_1' - 1 \end{bmatrix} \Delta x_{\tau-1} + \begin{bmatrix} \Psi_1' \\ \Psi_1' - 1 \end{bmatrix} x_{\tau-2} + \Psi_2 \varepsilon_t, \quad \Psi_2 = [\Psi_2' \; \Psi_2'']'
\]  
(A.1)

We start analyzing the second equation in (A.1):

\[
\Delta x_t = (\Psi_1' - 1) \Delta x_{\tau-1} + (\Psi_1' - 1) x_{\tau-2} + \Psi_2' \varepsilon_t
\]  
(A.2)

From the POR \( i_t = H_1 x_{\tau-1} + H_2 \varepsilon_t \) we get \( x_{\tau-2} = \frac{1}{H_1} i_{\tau-1} - \frac{H_2}{H_1} \varepsilon_{\tau-1} \), where \( H_1 \) is a scalar and \( H_2 \) is a vector.

Substituting into (A.2): \( \Delta x_t = (\Psi_1' - 1) \Delta x_{\tau-1} + \frac{\Psi_1' - 1}{H_1} i_{\tau-1} - \frac{(\Psi_1' - 1) H_2}{H_1} \varepsilon_{\tau-1} + \Psi_2' \varepsilon_t \)

Now we exploit the ALM of the economy \( z_t = \Psi_2' x_{\tau-1} + \Psi_2 \varepsilon_t \) to derive the vector of past shocks \( \varepsilon_{\tau-1} \), that is \( \varepsilon_{\tau-1} = \Psi_2' z_{\tau-1} - \Psi_2 \Psi_2' x_{\tau-2} \).

Now we add and subtract \( \Psi_2' \Psi_2' x_{\tau-2} \) to the previous equation:

\[
\varepsilon_{\tau-1} = \Psi_2' z_{\tau-1} - \Psi_2' \Psi_2' x_{\tau-2} + \Psi_2' \Psi_2' x_{\tau-2} - \Psi_2' \Psi_2' x_{\tau-2} = \Psi_2' z_{\tau-1} + \Psi_2' \Delta x_{\tau-2} - \Psi_2' \Psi_2' x_{\tau-2}
\]  
(A.3)

Now we have:

\[
\Delta x_t = (\Psi_1' - 1) \Delta x_{\tau-1} + \frac{\Psi_1' - 1}{H_1} i_{\tau-1} - \frac{(\Psi_1' - 1) H_2}{H_1} \varepsilon_{\tau-1} + \Psi_2' \varepsilon_t
\]

After some algebra on the previous equation:

\[
\Delta x_t = (\Psi_1' - 1) \Delta x_{\tau-1} + \frac{\Psi_1' - 1}{H_1} i_{\tau-1} - \frac{(\Psi_1' - 1) H_2}{H_1} \Psi_2' z_{\tau-1} - \frac{(\Psi_1' - 1) H_2}{H_1} \Psi_2' \Delta x_{\tau-2} + \frac{(\Psi_1' - 1) H_2}{H_1} \Psi_2' \Psi_2' x_{\tau-2} + \Psi_2' \varepsilon_t
\]

Continue with simplifications:
$$\Delta x = [\Psi^t - 1 - \frac{(\Psi^t - 1)H_t}{H_1} \Psi^t \Omega^t] \Delta x_{t-1} + \frac{\Psi^t - 1}{H_1} i_{t-1} - \frac{(\Psi^t - 1)H_t}{H_1} \Psi^t \Omega^t \left[ \pi_{t-1} \right] x_{t-1} + (\Psi^t - 1)H_t \Psi^t \Omega^t \Omega^t \varepsilon_t$$

Define $R_s = \frac{(\Psi^t - 1)H_t}{H_1} \Psi^t = [R^t_s \ R^t_s]$, so we have:

$$\Delta x = [\Psi^t - 1 - R_s \Psi^t] \Delta x_{t-1} + \frac{\Psi^t - 1}{H_1} i_{t-1} - \frac{R^t_s \pi_{t-1}}{H_1} + R_s \Psi^t \Omega^t \Omega^t \varepsilon_t$$

Finally:

$$\Delta x = [\Psi^t - 1 - R_s \Psi^t] \Delta x_{t-1} + \frac{\Psi^t - 1}{H_1} i_{t-1} - R^t_s \pi_{t-1} + (R_s \Psi^t - R^2_s) x_{t-1} + \Psi^t \varepsilon_t$$

(A.4)

Now we consider the inflation equation from (A.1) and in the same manner:

$$\pi = [\Psi^t - R_s \Psi^t] \Delta x_{t-1} + \frac{\Psi^t - R^t_s}{H_1} i_{t-1} - R^t_s \pi_{t-1} + (R_s \Psi^t - R^2_s) x_{t-1} + \Psi^t \varepsilon_t$$

(A.5)

where $R_s = \frac{\Psi^t H_s}{H_1} \Psi^t = [R^t_s \ R^t_s]$. 

In compact form, (A.4–A.5) could be rewritten as:

$$\begin{pmatrix} \pi_t \\ \Delta x_t \end{pmatrix} = \begin{pmatrix} -R^t_s \\ -R^t_s \end{pmatrix} \pi_{t-1} + \begin{pmatrix} \Psi^t - R_s \Psi^t \\ \Psi^t - 1 - R_s \Psi^t \end{pmatrix} \Delta x_{t-1} + \begin{pmatrix} \Psi^t \\ \Psi^t - 1 - R_s \Psi^t \end{pmatrix} i_{t-1} - \begin{pmatrix} R_s \Psi^t - R^2_s \\ R_s \Psi^t - R^2_s \end{pmatrix} x_{t-1} + \Psi^t \varepsilon_t.$$  

(A.6)

**Appendix B**

**The solution of the economy under the EHR and the VAR model of the public**

Here we derive the solution of the economy when the public forms its expectations using the VAR model, and the CB utilizes the EHR. Plugging the EHR in (10) into the state space of the economy in (3), we get:

$$\begin{pmatrix} \pi_t \\ x_t \end{pmatrix} = \begin{pmatrix} \alpha k + \beta \\ \alpha \end{pmatrix} E_{\pi_{t+1}} + \frac{1}{\alpha} \begin{pmatrix} \pi_{t+1} \\ x_{t+1} \end{pmatrix} \left( 1 + \mu E_{\pi_{t+1}} + \frac{1}{\alpha} E_{x_{t+1}} + \frac{k}{\alpha (k^2 + \lambda)} \varepsilon_t + \frac{1}{\alpha} \varepsilon_t \right) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \mu \\ \varepsilon_t \end{pmatrix}$$

where $\mu = \frac{\beta k}{\alpha (k^2 + \lambda)}$.

After some algebra we get:
\[
\begin{bmatrix}
\pi_t \\
x_t
\end{bmatrix} = \begin{bmatrix}
\alpha k + \beta k \\
\alpha
\end{bmatrix} E_i \begin{bmatrix}
\pi_{i+1} \\
x_{i+1}
\end{bmatrix} + \begin{bmatrix}
-\kappa \alpha (1+\beta) E_i \pi_{i+1} - k E_i x_{i+1} - \frac{k^2}{k^2 + \lambda} \varepsilon_t^\pi - k e_t^\pi \\
-\alpha (1+\beta) E_i \pi_{i+1} - E_i x_{i+1} - \frac{k}{k^2 + \lambda} \varepsilon_t^\pi - \varepsilon_t^x + e_t^x
\end{bmatrix} + \begin{bmatrix}
1 & k \\
0 & 1
\end{bmatrix} \begin{bmatrix}
\varepsilon_t^\pi \\
\varepsilon_t^x
\end{bmatrix}
\]

Continuing with simplifications:

\[
\begin{bmatrix}
\pi_t \\
x_t
\end{bmatrix} = \begin{bmatrix}
(\alpha k + \beta) E_i \pi_{i+1} + k E_i x_{i+1} - \kappa \alpha (1+\beta) E_i \pi_{i+1} - k E_i x_{i+1} - \frac{k^2}{k^2 + \lambda} \varepsilon_t^\pi - k e_t^\pi + e_t^x + k e_t^\pi \\
\alpha E_i \pi_{i+1} + E_i x_{i+1} - \alpha (1+\beta) E_i \pi_{i+1} - E_i x_{i+1} - \frac{k}{k^2 + \lambda} \varepsilon_t^\pi - e_t^x + e_t^x
\end{bmatrix}
\]

After rearrangement we get the dynamics of the economy implied by the EHR:

\[
\pi_t = (\beta - \kappa \alpha \mu) E_i \pi_{i+1} + \frac{\lambda}{k^2 + \lambda} \varepsilon_t^\pi, \quad (B.1)
\]

\[
x_t = -\alpha \mu E_i \pi_{i+1} - \frac{k}{k^2 + \lambda} \varepsilon_t^x. \quad (B.2)
\]

Notice that (B.1) and (B.2) hold for any type of expectations. In order to represent these equations in reduced form solution, we need first to express \( E_i \pi_{i+1} \) in terms of current shocks and past variables. To do so we substitute (B.1) and (B.2) into the VAR model of the public (taking the VAR one period ahead and applying conditional expectations for \( t+1 \)):

\[
E_i \pi_{i+1} = A_i[(\beta - \kappa \alpha \mu) E_i \pi_{i+1} + \frac{\lambda}{k^2 + \lambda} \varepsilon_t^\pi] + A_i[-\alpha \mu E_i \pi_{i+1} - \frac{k}{k^2 + \lambda} \varepsilon_t^x]
\]

\[
- A_i x_{i+1} + A_i[(1+\mu) E_i \pi_{i+1} + \frac{1}{\alpha} E_i x_{i+1} + \frac{k}{\alpha(k^2 + \lambda)} \varepsilon_t^x + \frac{1}{\alpha} \varepsilon_t^x]
\]

\[
E_i x_{i+1} = B_i[(\beta - \kappa \alpha \mu) E_i \pi_{i+1} + \frac{\lambda}{k^2 + \lambda} \varepsilon_t^\pi] + (1+B_i)[-\alpha \mu E_i \pi_{i+1} - \frac{k}{k^2 + \lambda} \varepsilon_t^x]
\]

\[
- B_i x_{i+1} + B_i[(1+\mu) E_i \pi_{i+1} + \frac{1}{\alpha} E_i x_{i+1} + \frac{k}{\alpha(k^2 + \lambda)} \varepsilon_t^x + \frac{1}{\alpha} \varepsilon_t^x]
\]

After some simplifications:
\[ E, \pi_{i+1} = A_i (\beta - \kappa \mu) E, \pi_{i+1} + A_i \frac{\lambda}{k^2 + \lambda} \varepsilon_i^\pi - A_i \alpha \mu E, \pi_{i+1} - A_i \frac{k}{k^2 + \lambda} \varepsilon_i^\pi \\
- A_i x_{i+1} + A_i (1 + \mu) E, \pi_{i+1} + A_i \frac{1}{\alpha} E, x_{i+1} + A_i \frac{k}{\alpha (k^2 + \lambda)} \varepsilon_i^\pi + A_i \frac{1}{\alpha} \varepsilon_i^x \\
E, x_{i+1} = B_i (\beta - \kappa \mu) E, \pi_{i+1} + B_i \frac{\lambda}{k^2 + \lambda} \varepsilon_i^\pi - (1 + B_2) \alpha \mu E, \pi_{i+1} - (1 + B_2) \frac{k}{k^2 + \lambda} \varepsilon_i^\pi \\
- B_2 x_{i+1} + B_2 (1 + \mu) E, \pi_{i+1} + B_2 \frac{1}{\alpha} E, x_{i+1} + B_2 \frac{k}{\alpha (k^2 + \lambda)} \varepsilon_i^\pi + B_2 \frac{1}{\alpha} \varepsilon_i^x \]

Continue with the simplifications:

\[(1 - A_i (\beta - \kappa \mu) + A_i \alpha \mu - A_i (1 + \mu) ) E, \pi_{i+1} - A_i \frac{1}{\alpha} E, x_{i+1} = \\
[A_i \frac{\lambda}{k^2 + \lambda} - A_i \frac{k}{k^2 + \lambda} + A_i \frac{k}{\alpha (k^2 + \lambda)} \varepsilon_i^\pi + A_i \frac{1}{\alpha} \varepsilon_i^x - A_i x_{i+1} \\
- B_i (\beta - \kappa \mu) + (1 + B_2) \alpha \mu - B_i (1 + \mu)] E, \pi_{i+1} + (1 - B_2 \frac{1}{\alpha}) E, x_{i+1} = \\
[B_i \frac{\lambda}{k^2 + \lambda} - (1 + B_2) \frac{k}{k^2 + \lambda} + B_i \frac{k}{\alpha (k^2 + \lambda)} \varepsilon_i^\pi + B_i \frac{1}{\alpha} \varepsilon_i^x - B_i x_{i+1} \]

Now the expectations could be expressed in the compact matrix form:

\[ G_i E, [\pi_{i+1}, x_{i+1}] = G_2 [\varepsilon_i^\pi, \varepsilon_i^x] + G_3 x_{i+1} , \]

where:

\[ G_i = \begin{bmatrix}
1 - A_i (\beta - \kappa \mu) + A_i \alpha \mu - A_i (1 + \mu) & -A_i \frac{1}{\alpha} \\
- B_i (\beta - \kappa \mu) + (1 + B_2) \alpha \mu - B_i (1 + \mu) & 1 - B_i \frac{1}{\alpha}
\end{bmatrix} \]

\[ G_2 = \begin{bmatrix}
A_i \frac{\lambda}{k^2 + \lambda} - A_i \frac{k}{k^2 + \lambda} + A_i \frac{k}{\alpha (k^2 + \lambda)} & A_i \frac{1}{\alpha} \\
B_i \frac{\lambda}{k^2 + \lambda} - (1 + B_2) \frac{k}{k^2 + \lambda} + B_i \frac{k}{\alpha (k^2 + \lambda)} & B_i \frac{1}{\alpha}
\end{bmatrix} \]

\[ G_3 = \begin{bmatrix}
-A_i \\
-B_2
\end{bmatrix} \]

Solving for \( E, \pi_{i+1} \) and \( E, x_{i+1} \):

\[ E, [\pi_{i+1}, x_{i+1}] = L [\varepsilon_i^\pi, \varepsilon_i^x] + K x_{i-1} , \text{ where } L = G_i^{-1} G_2 , \ K = G_i^{-1} G_3 . \] \hspace{1cm} (B.3)

For convenience we partition the matrices as \( L = \begin{bmatrix} L_i^{\pi} & L_i^{x \pi} \\ L_i^{x \pi} & L_i^{x x} \end{bmatrix} \) and \( K = \begin{bmatrix} K_i^\pi \\ K_i^x \end{bmatrix} . \hspace{1cm} (B.4) \]
Since the expectations for the output gap do not play any role in the dynamics of the economy (see (B.1) and (B.2)), we focus only on the inflation expectations. From (B.3) applying (B.4) we get:

\[ E_t x_{t+1} = L^{(n)}_x \tilde{\pi}_i^{n} + L^{(n)}_\pi \pi_i^{(n)} + K_x x_{t-1} . \]  

(B.5)

Substituting (B.5) into (B.1) and (B.2) we derive the solution of the economy:

\[ \pi_t = (\beta - \kappa \alpha \mu) [L^{(n)}_x \pi_i^{n} + L^{(n)}_\pi \pi_i^{(n)} + K_x x_{t-1}] + \frac{\lambda}{k + \lambda} \pi_i^{n}, \]

\[ x_t = -\alpha \mu [L^{(n)}_x \pi_i^{n} + L^{(n)}_\pi \pi_i^{(n)} + K_x x_{t-1}] - \frac{k}{k + \lambda} \pi_i^{n} \]

or

\[ \pi_t = [(\beta - \kappa \alpha \mu) L^{(n)}_x + \frac{\lambda}{k + \lambda}] \pi_i^{n} + (\beta - \kappa \alpha \mu) L^{(n)}_\pi \pi_i^{(n)} + (\beta - \kappa \alpha \mu) K_x x_{t-1} , \]

\[ x_t = -[\alpha \mu L^{(n)}_x + \frac{k}{k + \lambda}] \pi_i^{n} - \alpha \mu L^{(n)}_\pi \pi_i^{(n)} - \alpha \mu K_x x_{t-1} \]

In matrix form:

\[ z_t = \Phi_1 x_{t-1} + \Phi_2 \pi_t, \]

(B.6)

where

\[ \Phi_1 = \begin{bmatrix} (\beta - \kappa \alpha \mu) L^{(n)}_\pi \\ -\alpha \mu K_x \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} (\beta - \kappa \alpha \mu) L^{(n)}_x + \frac{\lambda}{k + \lambda} \\ -[\alpha \mu L^{(n)}_x + \frac{k}{k + \lambda}] \\ -\alpha \mu L^{(n)}_\pi \end{bmatrix}, \]

\[ \pi_t = \begin{bmatrix} \pi_i^{n} \\ \pi_i^{(n)} \end{bmatrix}, \quad x_t = \begin{bmatrix} x_i^{n} \\ x_i^{(n)} \end{bmatrix}. \]

Appendix C

Convergence of the economy under POR and EHR

The main question is whether the economy will converge to the MSV solution of REE, when agents are learning with decreasing gain and the CB uses POR instead of EHR. First, we present the MSV solution of REE:

\[ \pi_t = P_1 \pi_t^{n}, \quad x_t = P_2 \pi_t^{(n)}, \quad \text{where} \quad P_1 = \frac{\lambda}{k + \lambda}, \quad P_2 = \frac{k}{k + \lambda}. \]
It is useful to rewrite the solution for the output gap in terms of the same variables as in the VAR model:

$$\pi_t = P_t \sigma_t^\pi, \Delta x_t = P_x \sigma_x^\pi - \frac{P_t}{P_x} \pi_{t-1} \text{ (by exploiting } \sigma_{t-1}^\pi = \frac{1}{P_t} \pi_{t-1} \text{).}$$  \hspace{1cm} (C.1)

To find the convergence point of the economy, we should examine the ALM of the economy in (17) with the PLM of the public in (18). Appendix A represented the ALM of the economy by the same variables as in the PLM of the public. We write it here again:

$$
\begin{pmatrix} \pi_t \\ \Delta x_t \end{pmatrix} = \begin{bmatrix} -R_{\pi}^1 & R_{\pi}^2 \\ -R_x^1 & -R_x^2 \end{bmatrix} \pi_{t-1} + \begin{bmatrix} \Psi_{\pi} - R_{\pi} \Psi_t \\ \Psi_x - 1 \end{bmatrix} \Delta x_{t-1} + \begin{bmatrix} \Psi_{\pi} \\ \Psi_x - 1 \end{bmatrix} \varepsilon_t + \begin{bmatrix} R_{\pi} \Psi_t - R_x^2 \\ R_x \Psi_t - R_x \Psi_x \end{bmatrix} \varepsilon_{t-1} + \Psi \varepsilon_t,
\end{align*}
\hspace{1cm} (C.2)

where \( R_{\pi} = \frac{\Psi_{\pi} H_2}{H_1} \)  \( \Psi_x^{-1} = \begin{bmatrix} R_{\pi}^1 & R_{\pi}^2 \end{bmatrix} \) and \( R_x = \frac{(\Psi_x - 1) H_2}{H_1} \) \( \Psi_x^{-1} = \begin{bmatrix} R_x^1 & R_x^2 \end{bmatrix} \).

Note that all the parameters in (C.2) are determined both by the structural parameters of the CB and by the parameters of the VAR model of the public.\(^{14}\)

It is convenient to rewrite the PLM of the public from (6) as:

$$
\begin{pmatrix} \pi_t \\ \Delta x_t \end{pmatrix} = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \pi_{t-1} + \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} \Delta x_{t-1} + \begin{bmatrix} A_3 \\ B_3 \end{bmatrix} \varepsilon_t + \varepsilon_{t-1} \hspace{1cm} (C.3)
\end{align*}

The PLM of the public in (C.3) is under-parameterized relative to (C.2) because it excludes \( x_{t-1} \). However, the parameters in (C.2) are mutually dependent, that is, exclusion of one variable will automatically lead to exclusion of other variables, like a domino effect. For example, if \( \Psi_t = 0 \), then all parameters in the inflation dynamics in (C.2) will be equal to zero.

\(^{14}\) The definitions of all parameters are provided by Appendix A.
The fixed point of the differential equations in (C.4) below (for inflation and output gap, respectively) determines the convergence point of the economy:

\[
\begin{align*}
\frac{d(A, A_1, A_2)}{d\tau} &= \begin{bmatrix} -R^1_x - A_1 \\ \Psi' - R_x \Psi_1 - A_2 \\ H_1 - A_3 \end{bmatrix} \\
\text{and } \frac{d(B, B_2)}{d\tau} &= \begin{bmatrix} -R^i_y - B_i \\ \Psi' - 1 - R_x \Psi_1 - B_2 \\ H_1 - B_3 \end{bmatrix},
\end{align*}
\]  

(C.4)

where \( R_x, R_y, \Psi, \Psi_1, H_1, H_2 \) are implicit functions of the parameters \( A_j, B_j \) \( (i=1,2,3) \) in the VAR.

As it is impossible to derive the explicit analytical solution for (C.4), we find the convergence point numerically. The numerical results show that the economy converges to the MSV solution under RE shown in (C.1), such that the convergence point of the estimated parameters in the VAR are given by:

\( A_j = 0, \ (j=1,2,3) \ B_1 = -\frac{P_2}{P_1}, \ B_2 = B_3 = 0 \). This implies that the parameters of the ALM in (C.2) converge to \( \Psi_1 \rightarrow 0, \Psi_2 = \begin{bmatrix} \Psi_{21}^* \\ \Psi_{22}^* \end{bmatrix} \rightarrow \begin{bmatrix} P_1 \\ 0 \end{bmatrix} \).

Once the CB exploits EHR, the methodology here is the same as with the POR: we obtain the fixed point of the ALM and the PLM numerically and find that asymptotically the inflation dynamic will be implied by the REE, such that

\( A_j = 0, \ (j=1,2,3) \ B_1 = -\frac{P_2}{P_1}, \ B_2 = B_3 = 0 \). This implies that the parameters of the ALM in (C.2) converge to \( \Phi_1 \rightarrow 0, \Phi_2 = \begin{bmatrix} \Phi_{21}^* \\ \Phi_{22}^* \end{bmatrix} \rightarrow \begin{bmatrix} P_1 \\ 0 \end{bmatrix} \).
Appendix D
Survey of professional forecasters in Israel

In order to support our assumption of discrepancy in models between the CB and
the public, we conducted a survey among ten professional forecasters in Israel.
These professional forecasters provide to the Bank of Israel with their projections
for main economic variables on a monthly basis. They also attend regular meetings
(once a quarter) with the Governor and the Bank's Monetary Committee to share
their opinions concerning the state of the Israeli and foreign economies.
We present the survey questionnaire for the professional forecasters and briefly
explain the main answers obtained.

Questionnaire for professional forecasters in Israel

**Explanation:** This short questionnaire is designated for a research needs.

The questionnaire is completely anonymous.

For the questionnaire to be effective, please answer all listed questions correctly.

Thank you for your cooperation.

**Question 1:** Which main macroeconomic variables do you forecast?

**Question 2:** Are your forecasts derived from a model? If yes, is it structural or
statistical?

**Question 3:** If you have a model, does it contain the variable "output gap"? If yes,
how is it defined and measured?

**Question 4:** How much do you rely on projections of the Israeli Central Bank in your
own projections? (1-not at all, 5- very much). If you don't rely **very much** on the CB
projections, please explain why?

**Question 5:** How much are your forecasts typically different from the CB’s forecasts?
(1 – not different at all, 5 – very different)
Question 6: Have you ever used one of the models posted on the website of the Israeli Central Bank? If yes, which model have you used? If not, please explain why.

The main results
The most striking result is that none of the professional forecasters currently use, or had ever used in the past, the models of the BOI. Some professional forecasters pointed out that they would have liked to use the BOI's models but that they find these models to be very complicated and hardly operational. One professional forecaster claimed that he (and his team) found serious flaws in the models of the CB. The professional forecasters rely on the projections of the BOI to a moderate extent. The professional forecasters’ projections differ from BOI projections to a moderate extent. Only one professional forecaster uses the "output gap" in his model, whereas the majority uses the growth rate of output.